Topographic Properties Process: Computation of Slope, Aspect, and Curvature

Slope, aspect, and curvature are computed for each cell in the DEM from the coefficients of a quadratic polynomial function relating elevation **z** to the plan cartesian coordinates **x** and **y** over a 3 x 3 neighborhood centered on the current cell: (notation from Evans, 1980, reference given subsequently)

$\mathbf{z} = a\mathbf{x}^2 + b\mathbf{y}^2 + c\mathbf{xy} + d\mathbf{x} + e\mathbf{y} + f$

where

a = dxx, second derivative in x direction
b = dyy, second derivative in y direction
c = dxy
d = dx, first derivative in x direction
e = dy, first derivative in y direction

slope (deg) = $\arctan((d^2 + e^2)^{1/2})$;   slope (pct) = $100 * (d^2 + e^2)^{1/2}$

aspect (deg) = $180 - \arctan(e / d) + 90*(d / |d|)$

profile curvature = $-2 * (ad^2 + be^2 + cde) / (1 + d^2 + e^2)^{1.5}$

plan curvature = $-2 * (bd^2 + ae^2 - cde) / (d^2 + e^2)^{1.5}$

The coefficients *a* through *e* are computed as finite differences from the 3 x 3 neighborhood. The computation of these coefficients can be varied so that the quadratic polynomial represents different methods of fitting a mathematically-described elevation surface to the z values in the 3 x 3 neighborhood.

The following cell notation is used in the subsequent descriptions (where z5 is current cell):

z1	z2	z3
z4	z5	z6
z7	z8	z9

cell_x, cell_y: cell size in column and line direction, respectively

All methods below compute coefficient *c* in the same way:

c = (z3 + z7 – z1 – z9) / (4 * cell_x * cell_y)

Surface-fitting methods for the 3 x 3 window:

1) Exact fit to 4 nearest neighbors and center cell
2) Quadratic surface, least-squares fit
3) Quadratic surface, least-squares fit, weighted by 1/distance^2
4) Quadratic surface, least-squares fit, weighted by 1/distance
5) Quadratic surface, least-squares fit, match central cell

These surface-fitting methods differ in their computation of first derivatives (*c* and *d*) and second derivatives (*a* and *b*). Method 1 uses 5 cell values to compute 5 coefficients, producing a local surface that exactly fits the center and 4 nearest cells with no smoothing. The other four methods use all nine cells in the neighborhood in computing first and second derivatives. Since this is more than the minimum number of values required for the computation, the resulting surfaces are "best-fit" smoothed surfaces that may not pass exactly through the elevation data values. Each of the derivatives in the x and y directions is the weighted average of 3 sets of finite elevation differences.

1) Exact fit to 4 nearest neighbors and center cell.

Zevenbergen, L.W., and Thorne C.R., (1987), Quantitative analysis of land surface topography. *Earth Surface Processes and Landforms*, 12:47-56.

Florinsky, Igor V., 1998, Accuracy of local topographic variables derived from digital elevation models. Int. J. Geographical Information Science, 12:47-61.

Schmidt, J., Evans, I.S., and Brinkmann, J., 2003, Comparison of polynomial models for land surface curvature calculation. Int. J. Geographical Information Science, 17: 797-814.

This method uses the center cell value (z5) and 4 nearest neighbors to compute the second derivative coefficients (*a* and *b*) that are involved in computation of curvature. First derivatives (*d* and *e*) are computed using only the 4 nearest neighbors.

a = (z4 + z6 – 2z5) / cell_x^2
b = (z2 + z8 - 2z5) / cell_y^2
d = (z6 – z4) / (2 * cell_x)
e = (z2 – z8) / (2 * cell_y)

2) Quadratic surface, least-squares fit

Evans, I.S., 1980, An integrated system of terrain analysis and slope mapping, *Zeitschrift fur Geomorphologie,* N.F., Supplement 36: 274-295.

Schmidt, J., Evans, I.S., and Brinkmann, J., 2003, Comparison of polynomial models for land surface curvature calculation. Int. J. Geographical Information Science, 17: 797-814.

This method produces a best-fit quadratic surface that is not constrained to pass through any of the elevations in the 3 x 3 window. The center cell and all eight neighbors are used to compute the second derivative coefficients. First derivatives are computed using the eight neighbor cell values. All cells in the neighborhood are weighted equally in these computations.

a = (z1 + z3 + z4 + z6 + z7 + z9 – 2*(z2 + z5 + z8)) / (6 * cell_x^2)
b = (z1 + z2 + z3 + z7 + z8 + z9 – 2*(z4 + z5 + z6)) / (6 * cell_y^2)
d = (z3 + z6 + z9 – z1 – z4 – z7) / (6 * cell_x)
e = (z1 + z2 + z3 – z7 – z8 – z9) / 6 * cell_y)

3) Quadratic surface, least-squares fit, weighted by 1/distance^2

Horn, B.K.P., 1981, Hill shading and the reflectance map. Proceedings of IEEE, 69:14-47.

Zhou, Qimig and Liu, Zuejun, 2004, Error analysis on grid-based slope and aspect algorithms. Photogrammetric Engineering & Remote Sensing 70: 957-962.

This method differs from method 2 in assigning greater weight to the 4 nearest neighbor cells than to the cells on the 4 corners of the neighborhood. Finite differences involving these nearest cells are counted twice in computing the averaged derivatives, equivalent to weighting by the inverse of the square of the distance from the center cell.

a = (z1 + z3 + 2*z4 + 2*z6 + z7 + z9 – 2*(z2 + 2*z5 + z8)) / (8 * cell_x^2)
b = (z1 + 2*z2 + z3 + z7 + 2*z8 + z9 – 2*(z4 + 2*z5 + z6)) / (8 * cell_y^2)
d = (z3 – z1 + 2(z6 – z4) **+** z9 – z7) / (8 * cell_x)
e = (z1 – z7 + 2(z2 – z8) + z3 – z9) / (8 * cell_y)

4) Quadratic surface, least-squares fit, weighted by 1/distance

Unwin, D., *Introductory Spatial Analysis*. Methuen, London and New York.

Zhou, Qiming and Liu, Zuejun, 2004, Error analysis on grid-based slope and aspect algorithms. Photogrammetric Engineering & Remote Sensing 70: 957-962.

Method 4 also weights the neighboring cell values by distance, multiplying each of these values by the square root of 2, equivalent to weighting by the inverse of the distance from the center cell.

a = (z1 + z3 + sqrt(2)*z4 + sqrt(2)*z6 + z7 + z9 –2*(z2 + sqrt(2)*z5 + z8)) /
((4 + 2*sqrt(2)) * cell_x^2)
b = (z1 + sqrt(2)*z2 + z3 + z7 + sqrt(2)*z8 + z9 –2*(z4 + sqrt(2)*z5 + z6)) /
((4 + 2*sqrt(2)) * cell_y^2)
d = (z3 – z1 + sqrt(2)*(z6 – z4) **+** z9 – z7) / ((4 + 2*sqrt(2)) * cell_x)
e = (z1 – z7 + sqrt(2)*(z2 – z8) + z3 – z9) / ((4 + 2*sqrt(2)) * cell_y)

5) Quadratic surface, least-squares fit, match central cell

Shary, P.A., 1995, Land surface in gravity points classification by complete system of curvatures. Mathematical Geology, 27: 373-390.

Florinsky, Igor V., 1998, Accuracy of local topographic variables derived from digital elevation models. Int. J. Geographical Information Science, 12:47-61.

Schmidt, J., Evans, I.S., and Brinkmann, J., 2003, Comparison of polynomial models for land surface curvature calculation. Int. J. Geographical Information Science, 17: 797-814.

For purposes of computing curvature, this method uses additional weighting for the center cell to force the computed surface to pass through the current cell's elevation value. First derivatives for the slope re computed as for method 2.

a = (z1 + z3 + z7 + z9 + 3*(z4 + z6) – 2*(z2 + 3*z5 + z8)) / (10 * $cell\_x^2$)
b = (z1 + z3 + z7 + z9 + 3*(z2 + z8) – 2*(z4 + 3*z5 + z6)) / (10 * $cell\_y^2$)
d = (z3 + z6 + z9 – z1 – z4 – z7) / (6 * cell_x)
e = (z1 + z2 + z3 – z7 – z8 – z9) / 6 * cell_y)